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Pore-water pressure as the stress reference in the stress state variables for unsaturated soils, a theoretical revisit

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Abstract
Stress state variables refer to the state of stresses that control shear strength and volume change of unsaturated soils. When the pore-water pressure, u_w, is taken as the stress reference, the stress state variables are expressed as $(\sigma - u_w)$, $(u_a - u_w)$, and u_w for soils with compressible soil solids, and as $(\sigma - u_w)$ and $(u_a - u_w)$ for soils with incompressible soil solids. The existing theoretical derivation was based on the equilibrium equation of water, air, and contractile skin phases. However, several steps in the existing derivation are not clearly presented and are challenging to follow. These steps are in the initial stage of the equilibrium derivation and in the elimination of the third variable, u_w, for incompressible soil solids. This paper presents revisit of the theoretical derivation to obtain the stress state variables. This revisit considers stress times area in place of force instead of stress times porosity in place of force as in the existing derivation. The proposed approach provides a clearer and more transparent procedure for eliminating u_w from the governing equations when the solid particles are assumed to be incompressible. This revisit also provides sound stress state variables for unsaturated soils with both compressible and incompressible solids. Overall, the theoretical revisit in this paper offers clearer interpretation and a more comprehensive derivation. In addition, it supports the validity and correctness of the commonly adopted stress state variables used in the constitutive modelling of unsaturated soils.

Keywords:
Pore-water Pressure;
Stress State Variable;
Theoretical Soil Mechanics;
Total Normal Stress;
Unsaturated Soils;

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INTRODUCTION

Stress state variables are the state of stresses in a soil that control shear strength and volume change of unsaturated soils. All the formulations of the shear strength and volume change of unsaturated soils use the stress state variables. The stress state variables of unsaturated soils -using the pore-air pressure, u_a, as the stress reference- are the net normal stress, $(\sigma - u_a)$, and the matric suction, $(u_a - u_w)$ [1][2]. This means that the stresses are quantified as the difference with the pore-air pressure, u_a. These stress state variables are used in several analyses, as in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The shear strength equation that uses these stress state variables is [1,2 13,14]:

$$\tau_{ff} = c' + (\sigma_f - u_a)_f \tan\phi' + (u_a - u_w)_f \tan\phi^b \quad (1)$$

where τ_{ff} is the shear stress at failure on the failure plane, $(\sigma_f - u_a)_f$ is the net normal stress at failure on the failure plane, $(u_a - u_w)_f$ is the matric suction at failure, and ϕ^b is the angle that quantifies the increase in shear strength due to the increase in matric suction. The shear strength (1) is used in

[1, 2, 19, 20, 21, 22, 23, 24, 25, 26]. The volume change equations that use these stress state variables are [1]:

$$de = a_t d(\sigma - u_a) + a_m d(u_a - u_w) \quad (2)$$

and

$$dw = b_t d(\sigma - u_a) + b_m d(u_a - u_w) \quad (3)$$

where e is the void ratio, w is the water content, a_t is the coefficient of compressibility with respect to a change in net normal stress, a_m is the coefficient of compressibility with respect to a change in matric suction, b_t is the coefficient of water content change with respect to a change in net normal stress, and b_m is the coefficient of water content change with respect to a change in matric suction. The volume change of unsaturated soils in (2) and (3) are used in [26, 27, 28].

When the pore-water pressure, u_w, is taken as the stress reference, the stress state variables are expressed as $(\sigma - u_w)$, $(u_a - u_w)$, and u_w for compressible soil solids, $(\sigma - u_w)$ and $(u_a - u_w)$ for incompressible soil solids. An application of the use of the stress state variables using the pore-water pressure, u_w, as the stress reference is in the extended Mohr-Coulomb failure envelope equation in $(\sigma - u_w)$ and $(u_a - u_w)$ [2]:

$$\tau = c' + (\sigma - u_w)\tan\phi' + (u_a - u_w)\tan\phi'' \quad (4)$$

where c' is the effective cohesion, ϕ' is the effective friction angle with respect to changes in $(\sigma - u_w)$ when $(u_a - u_w)$ is held constant, and ϕ'' is the angle with respect to changes in $(u_a - u_w)$ when $(\sigma - u_w)$ is held constant. An example of a plot of the extended Mohr-Coulomb equation (4) using the stress state variables $(\sigma - u_w)$ and $(u_a - u_w)$ is shown in Figure 1. The volume change equations that use this stress variables can be formulated as:

$$de = f\left(d(\sigma - u_a), d(u_a - u_w)\right) \quad (5)$$

and

$$dw = f\left(d(\sigma - u_a), d(u_a - u_w)\right) \quad (6)$$

However, there is a limitation in the theoretical derivation available in the literature as explained in the literature review section of this paper. This existing derivation of the stress used the porosity with respect to the water phase, n_w, in the term $-n_w\left(u_w + \frac{\partial u_w}{\partial y}dy\right)dx.dz + n_w u_w dx.dz$ as in (13). In addition, the previous derivation did not clearly explain the condition of compressible and incompressible soil solids.

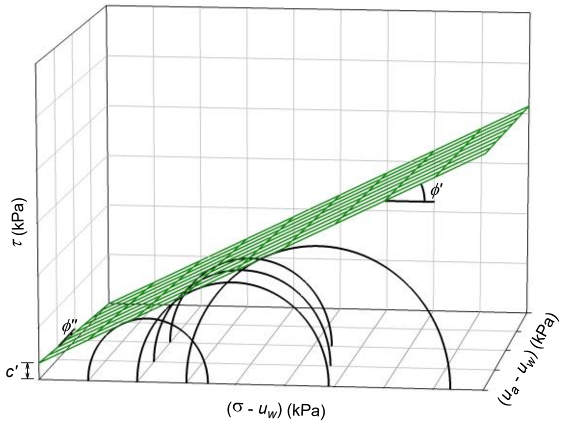


Figure 1. Example Plot of the Extended Mohr-Coulomb Failure Envelope Using the Stress State Variables $(\sigma - u_w)$ and $(u_a - u_w)$

The elimination of u_w in the term $(n_c + n_s)\partial u_w/\partial y$ as the soil solids are incompressible is difficult to follow. As the stress state variables control the shear strength and the volume change of unsaturated soils, the correctness of the derivation of the stress state variables is therefore essential. This is because the validity of the analysis of unsaturated shear strength and unsaturated volume change depends on the correctness of the derivation of the stress state variables for unsaturated soils. Thus, a revisit of the derivation is necessary.

This paper presents a detailed derivation of the stress state variables of unsaturated soils using the pore-water pressure, u_w, as the stress reference. The stress state variables are derived for unsaturated soils with compressible and incompressible soil solids. A new consideration of the equilibrium equation and the compressibility of soil solids as that in the existing theoretical derivation is introduced and utilized.

METHOD

The study is mainly a theoretical derivation. The flowchart of the methodology is shown in Figure 2. The study was started by going through the literature of the derivation of the stress state variables for unsaturated soils. The fundamental derivation was put into emphasis. A gap was then obtained in the derivation of the stress state variables for unsaturated soils using pore-water pressure, u_w, as the stress reference. The limitation of the previous derivation in describing the effect of compressibility of soil solids is obtained. A theoretical derivation was performed that better addresses the gap of the previous derivation. The derivation uses the equilibrium equation in [1][29].

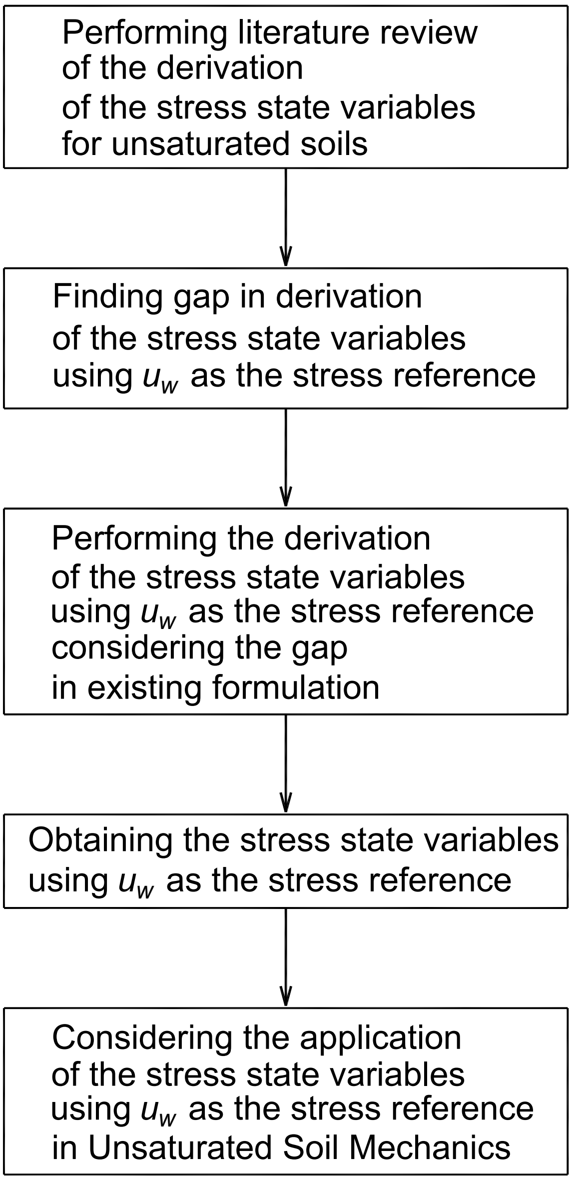


Figure 2. Flow Chart of the Research Methodology

The equilibrium equations for water, air, and matric suction were reformulated using the unit area of water phase, unit area of air phase, and unit area of contractile skin, respectively. The stress reference of pore water, u_w, was used. The compressibility of soil solids was considered using the parameter the additional cross-sectional area filled with solids per gross cross-sectional area of the soil element due to the compressibility of the soil solids. Using this methodology, the stress state variables of unsaturated soils were obtained, demonstrating a more rigorous derivation approach. Finally, the application aspect of the theoretical derivation from this study is presented in this paper.

LITERATURE REVIEW

The concept of stress state variable(s) is revisited, starting from the stress state variable of saturated soils, followed by the initial stage of stress state variables of unsaturated soils, and concluding with the last stage of the stress state variables of unsaturated soils. The stress state variable of saturated soils is the effective stress [30]:

$$\sigma' = \sigma - u_w \quad (7)$$

where σ' is the effective stress, σ is the total stress, and u_w is the pore-water pressure. This means that the shear strength and consolidation stress are quantified using the effective stress as the stress state variable. The shear strength of saturated soils is quantified as:

$$\tau = c' + \left(\sigma - u_w\right)\tan\phi' \quad (8)$$

The volume change of normally consolidated saturated soils (i.e., the consolidation settlement) is quantified as:

$$s_c = \frac{C_c H}{1+e_o}\log\left(\frac{\sigma'_{vo} + \Delta\sigma}{\sigma'_{vo}}\right) \quad (9)$$

where s_c is the magnitude of consolidation, C_c is the compression index, H is the thickness of the consolidated layer, σ'_{vo} is the initial effective vertical stress, $\Delta\sigma_v$ is the change in vertical effective stress, and e_o is the initial void ratio.

In the early-stage development of unsaturated soil mechanics, the stress state variable was formulated as [31]:

$$\sigma' = \left(\sigma - u_a\right) + \chi\left(u_a - u_w\right) \quad (10)$$

where χ is a parameter (a factor) that depends on soil type and degree of saturation, S. Equation (10) is inferior to (7) as in the effective stress equation, there is no factor that depends on soil type χ as in (10). A stress state variable(s) should be independent of the soil type [32][33].

To overcome this, Fredlund and Morgenstern [1] proposed two independent stress state variables for unsaturated soils. Using u_w as the reference, these stress state variables are:

$$\left(\sigma - u_w\right) \quad (11)$$

and

$$\left(u_a - u_w\right) \quad (12)$$

these stress state variables are used in the shear strength and the volume change equations in (5) and (6), respectively.

The proof explained in [1][3] is started with equilibrium equations for water, air, and contractile skin using Figure 3 for the axis orientation. For the water phase, the sum of water forces in the *y*-direction = 0 uses the following relation:

$$\begin{aligned}&-n_w\left(u_w+\frac{\partial u_w}{\partial y}dy\right)dx.dz+n_wu_wdx.dz\\&-n_w\rho_wgdx.dy.dz-F^w_{sy}dx.dy.dz\\&-F^w_{cy}dx.dy.dz=0\end{aligned} \quad (13)$$

where n_w is the porosity with respect to the water phase (14), ρ_w is the water density, g is the gravitational acceleration, F^w_{sy} is the interaction force between the water phase and the soil solids in the *y*-direction, F^w_{cy} is the interaction force between the water phase and the soil contractile skin in the *y*-direction. The parameter n_w is defined as [1][3]:

$$n_w=V_w/V \quad (14)$$

For the air phase, the sum of air forces in the *y*-direction = 0 uses the following relation:

$$\begin{aligned}&-n_a\left(u_a+\frac{\partial u_a}{\partial y}dy\right)dx.dz+n_au_adx.dz\\&-n_a\rho_agdx.dy.dz-F^a_{sy}dx.dy.dz\\&-F^a_{cy}dx.dy.dz=0\end{aligned} \quad (15)$$

where n_a is the porosity with respect to the air phase (16), ρ_a is the air density, F^a_{sy} is the interaction force between the air phase and the soil solids in the *y*-direction, F^a_{cy} is the interaction force between the air phase and the contractile skin in the *y*-direction. The parameter n_a is defined as [1][3]:

$$n_a=V_a/V \quad (16)$$

For the contractile skin phase, the sum of water forces in the *y*-direction = 0 uses the following relation:

$$\begin{aligned}&n_c\left(\begin{aligned}&\sigma^c_y+\frac{\partial f^*}{\partial y}(u_a-u_w)dy\\&+f^*\frac{\partial(u_a-u_w)}{\partial y}dy\end{aligned}\right)dx.dz-n_c\sigma^c_ydx.dz\\&-n_c\rho_cg+F^w_{cy}dx.dy.dz+F^a_{cy}dx.dy.dz=0\end{aligned} \quad (17)$$

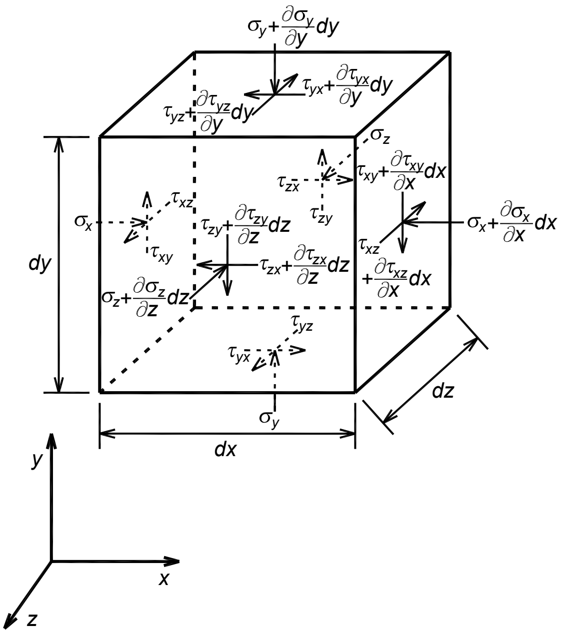


Figure 3. Soil Element and Axis Orientation in the Derivation of the Stress State Variables for Unsaturated Soils

where n_c is the porosity with respect to the contractile skin phase (18), ρ_c is the contractile skin density, σ^c_y is the normal stress in the contractile skin in the *y*-direction, f^* is the final interaction between the contractile skin and the soil structure equilibrium. The parameter n_c is defined as [1][3]:

$$n_c=V_c/V \quad (18)$$

Equations (13), (15), (17) were then incorporated in a structural equilibrium equation to obtain the equilibrium equation in the *y*-direction using the pore-water pressure, u_w as the stress reference is:

$$\begin{aligned}&\frac{\tau_{xy}}{\partial x}+\frac{\partial(\sigma_y-u_w)}{\partial y}-(n_a-n_cf^*)\frac{\partial(u_a-u_w)}{\partial y}\\&+\frac{\partial\tau_{zy}}{\partial z}+(n_c+n_s)\frac{\partial u_w}{\partial y}+n_s\rho_sg-F^w_{sy}\\&-F^a_{sy}+n_c(u_a-u_w)\frac{\partial f^*}{\partial y}=0\end{aligned} \quad (19)$$

Fredlund and Rahardjo [1] stated that the combinations of stress state variables from (19) are $(\sigma-u_w)$, (u_a-u_w), and u_w. These stresses are the stress state variables of unsaturated soils using the pore-water pressure, u_w, as the stress reference for soils with compressible soil solids. For unsaturated soils with incompressible soil solids, the term $(n_c+n_s)\partial u_w/\partial y$ can be eliminated from (19) [1][3]. Therefore, the stress state variables for unsaturated soils with incompressible soil solids using the pore-water pressure, u_w, as the stress reference are ($\sigma - u_w$) and ($u_a - u_w$).

The equilibrium in (13), (15), and (17) use the porosities, n (i.e., n_w, n_a, and n_c) times stress in place of force. This is difficult to follow. In addition, it is also difficult to follow that the fifth term in (19) (the term $(n_c + n_s)\partial u_w/\partial y$) is equal to zero as the soil solids are incompressible. Krisnanto et al. [29] found that the porosity with respect to the soil solid, n_s, is not zero for the typical values of the porosity, n. The range of the porosity, n, value is between 0.12 and 0.84. Thus, substituting the value of the porosity, n, into the following relationship:

$$n_s = 1 - n \tag{20}$$

results the porosity with respect to the soil solid, n_s, is not zero. This makes the fifth term in (19) (i.e., the term $(n_c + n_s)\partial u_w/\partial y$) is not zero. Consequently, the term u_w cannot be eliminated. Therefore, there is a need to provide a further derivation of (19) to account for the elimination of the term u_w for the unsaturated soils with incompressible soil solids.

The same condition as in the y- direction occurs in the x- and z- directions. For the water phase, the sum of water forces in the x-direction = 0 uses the following relation:

$$\begin{aligned} &-n_w\left(u_w + \frac{\partial u_w}{\partial x}dx\right)dy.dz + n_w u_w dy.dz \\ &-F_{sx}^w dx.dy.dz - F_{cx}^w dx.dy.dz = 0 \end{aligned} \tag{21}$$

where F_{sx}^w is the interaction force between the water phase and the soil solids in the x-direction, F_{cx}^w is the interaction force between the water phase and the contractile skin in the x-direction.

For the air phase, the sum of air forces in the x-direction = 0 uses the following relation:

$$\begin{aligned} &-n_a\left(u_a + \frac{\partial u_a}{\partial x}dx\right)dy.dz + n_a u_a dy.dz \\ &-F_{sx}^a dx.dy.dz - F_{cx}^a dx.dy.dz = 0 \end{aligned} \tag{22}$$

where F_{sx}^a is the interaction force between the air phase and the soil solids in the x-direction, F_{cx}^a is the interaction force between the air phase and the contractile skin in the x-direction.

For the contractile skin phase, the sum of contractile skin forces in the x-direction = 0 uses the following relation:

$$\begin{aligned} &n_c\begin{pmatrix} \sigma_x^c + \frac{\partial f^*}{\partial x}(u_a - u_w)dx \\ +f^*\frac{\partial(u_a - u_w)}{\partial x}dx \end{pmatrix}dy.dz - n_c\sigma_x^c dy.dz \\ &+F_{cx}^w dx.dy.dz + F_{cx}^a dx.dy.dz = 0 \end{aligned} \tag{23}$$

where σ_x^c is the normal stress in the contractile skin in the x-direction.

Equations (21) to (23) were then incorporated in a structural equilibrium equation. The final equilibrium equation is:

$$\begin{aligned} &\frac{\partial(\sigma_x - u_w)}{\partial x} - (n_a - n_c f^*)\frac{\partial(u_a - u_w)}{\partial y} + \frac{\partial \tau_{yx}}{\partial y} \\ &+\frac{\partial \tau_{zx}}{\partial z} + (n_c + n_s)\frac{\partial u_w}{\partial x} - F_{sx}^w - F_{sx}^a \\ &+n_c(u_a - u_w)\frac{\partial f^*}{\partial x} = 0 \end{aligned} \tag{24}$$

Similar to the y-direction, the equilibrium in (21) to (23) use the porosities, n (i.e., n_w, n_a, and n_c) times stress in place of force. As previously noted, this concept is difficult to follow. In addition, analogous to the y-direction derivation, it is difficult to follow that the fifth term in (24) (the term $(n_c + n_s)\partial u_w/\partial x$) is equal to zero as the soil solids are incompressible.

For the water phase, the sum of water forces in the z-direction = 0 uses the following relation:

$$\begin{aligned} &-n_w\left(u_w + \frac{\partial u_w}{\partial z}dz\right)dx.dy + n_w u_w dx.dy \\ &-F_{sz}^w dx.dy.dz - F_{cz}^w dx.dy.dz = 0 \end{aligned} \tag{25}$$

where F_{sz}^w is the interaction force between the water phase and the soil solids in the z-direction, F_{cz}^w is the interaction force between the water phase and the contractile skin in the z-direction.

For the air phase, the sum of air forces in the z-direction = 0 uses the following relation:

$$\begin{aligned} &-n_a\left(u_a + \frac{\partial u_a}{\partial z}dz\right)dx.dy + n_a u_a dx.dy \\ &-F_{sz}^a dx.dy.dz - F_{cz}^a dx.dy.dz = 0 \end{aligned} \tag{26}$$

where F_{sz}^a is the interaction force between the air phase and the soil solids in the z-direction, F_{cz}^a is the interaction force between the air phase and the contractile skin in the z-direction.

For the contractile skin phase, the sum of contractile skin forces in the z-direction = 0 uses the following relation:

$$n_c\begin{pmatrix}\sigma_z^c+\dfrac{\partial f^*}{\partial z}(u_a-u_w)dz\\+f^*\dfrac{\partial(u_a-u_w)}{\partial z}dz\end{pmatrix}dx.dy-n_c\sigma_z^c dx.dy \\ +F_{cz}^w dx.dy.dz+F_{cz}^a dx.dy.dz=0 \quad (27)$$

where σ_z^c is the normal stress in the contractile skin in the *z*-direction.

Equations (25) to (27) were then incorporated in a structural equilibrium equation. The final equilibirum equation is:

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial(\sigma_z-u_w)}{\partial z} \\ -(n_a-n_c f^*)\frac{\partial(u_a-u_w)}{\partial z}+(n_c+n_s)\frac{\partial u_w}{\partial z} \\ -F_{sz}^w-F_{sz}^a+n_c(u_a-u_w)\frac{\partial f^*}{\partial z}=0 \quad (28)$$

Likewise, the equilibrium in (25) to (27) use the porosities, *n* (i.e., n_w, n_a, and n_c) times stress in place of force. This is again difficult to follow. Furthermore, it is difficult to follow that the fifth term in (28) (the term $(n_c+n_s)\partial u_w/\partial z$) is equal to zero as the soil solids are incompressible.

THEORETICAL DERIVATION

Derivation in the *y*-Direction

The axis orientations (the *x*-, *y*-, and *z*-directions) for the derivation use those in Figure 3. The derivation in the *y*-direction is performed to obtain the equilibrium equation in the *y*-direction. In equilibrium, the sum of forces in the *y*-direction is equal to zero, thus:

$$\tau_{xy}dy.dz-\left(\tau_{xy}+\frac{\partial\tau_{xy}}{\partial x}dx\right)dy.dz \\ +\tau_{zy}dx.dy-\left(\tau_{zy}+\frac{\partial\tau_{zy}}{\partial z}dz\right)dx.dy \\ +\sigma_y dx.dz-\left(\sigma_y+\frac{\partial\sigma_y}{\partial y}dy\right)dx.dz \\ -\rho g dx.dy.dz=0 \quad (29)$$

where ρ is the density of soil. Continuing the derivation in (29), the equilibrium equation for an unsaturated soil element in the *y*-direction is:

$$\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+\rho g\right)dx.dy.dz=0 \quad (30)$$

The equilibrium equation for the water phase in the y-direction is:

$$-a_w\left(u_w+\frac{\partial u_w}{\partial y}dy\right)dx.dz+a_w u_w dx.dz \\ -n_w\rho_w g dx.dy.dz-F_{sy}^w dx.dy.dz \\ -F_{cy}^w dx.dy.dz=0 \quad (31)$$

where a_w is the cross-sectional area filled with water per gross cross-sectional area of the soil element. Continuing the derivation in (31), the equilibrium for the water phase in the *y*-direction is:

$$\left(a_w\frac{\partial u_w}{\partial y}+n_w\rho_w g+F_{sy}^w+F_{cy}^w\right)dx.dy.dz=0 \quad (32)$$

For the air phase, the sum of air forces in the *y*-direction = 0 uses the following relation:

$$-a_a\left(u_a+\frac{\partial u_a}{\partial y}dy\right)dx.dz+a_a u_a dx.dz \\ -a_a\rho_a g dx.dy.dz-F_{sy}^a dx.dy.dz \\ -F_{cy}^a dx.dy.dz=0 \quad (33)$$

where a_a is the cross-sectional area filled with air per gross cross-sectional area of the soil element. Continuing the derivation in (33), the equilibrium equation for the air phase in the *y*-direction is:

$$\left(a_a\frac{\partial u_a}{\partial y}+n_a\rho_a g+F_{sy}^a+F_{cy}^a\right)dx.dy.dz=0 \quad (34)$$

For the contractile skin phase, the sum of contractile skin forces in the *y*-direction = 0 uses the following relation:

$$a_c\begin{pmatrix}\sigma_y^c+\dfrac{\partial f^*}{\partial y}(u_a-u_w)dy\\+f^*\dfrac{\partial(u_a-u_w)}{\partial y}dy\end{pmatrix}dx.dz-a_c\sigma_y^c dx.dz \\ -a_c\rho_c g+F_{cy}^w dx.dy.dz+F_{cy}^a dx.dy.dz=0 \quad (35)$$

where a_c is the cross-sectional area filled with contractile skin per gross cross-sectional area of the soil element. Continuing the derivation in (35), the equilibrium equation for the contractile skin in the *y*-direction is:

$$\begin{pmatrix}-a_c(u_a-u_w)\dfrac{\partial f^*}{\partial y}\\-a_c f^*\dfrac{\partial(u_a-u_w)}{\partial y}\\+n_c\rho_c g-F_{cy}^w-F_{cy}^a\end{pmatrix}dx.dy.dz=0 \quad (36)$$

The equilibrium for the soil structure is calculated using the following relationship as in [1][29]:

$$\begin{aligned}&\text{total equilibrium of unsaturated soil element}\\&-\text{equilibrium for water phase}\\&-\text{equilibrium for air phase}\\&-\text{equilibrium for contractile skin}=0\end{aligned} \quad (37)$$

Substituting (30), (32), (34), and (36) into (37) results in the equilibrium equation for the soil structure in the y-direction:

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}-a_a\frac{\partial u_a}{\partial y}-a_w\frac{\partial u_w}{\partial y}\\&+a_cf^*\frac{\partial(u_a-u_w)}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\\&+(\rho-n_a\rho_a-n_w\rho_w-n_c\rho_c)g-F_{sy}^w-F_{sy}^a\\&+a_c(u_a-u_w)\frac{\partial f^*}{\partial y}=0\end{aligned} \quad (38)$$

From [1]:

$$n_a+n_w+n_c+n_s=1 \quad (39)$$

$$\rho=n_a\rho_a+n_w\rho_w+n_c\rho_c+n_s\rho_s \quad (40)$$

where ρ is the density of the soil, ρ_a is the density of air, ρ_w is the density of water, ρ_c is the density of contractile skin, and ρ_s is the density of soil solids. Equation (40) can be written in the following form:

$$n_s\rho_s=\rho-n_a\rho_a-n_w\rho_w-n_c\rho_c \quad (41)$$

Substituting (41) into (38):

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}-a_a\frac{\partial u_a}{\partial y}-a_w\frac{\partial u_w}{\partial y}\\&+a_cf^*\frac{\partial(u_a-u_w)}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+n_s\rho_sg\\&-F_{sy}^w-F_{sy}^a+a_c(u_a-u_w)\frac{\partial f^*}{\partial y}=0\end{aligned} \quad (42)$$

The gross cross-sectional area of the soil element is equal to 1. Thus:

$$a_a+a_w+a_c+a_s+\Delta a_s=1 \quad (43)$$

where a_s is the cross-sectional area filled with solids per gross cross-sectional area of the soil element and Δa_s is the change of the cross-sectional area filled with solids per gross cross-sectional area of the soil element due to the compressibility of the soil solids.

Using pore-water pressure, u_w, as the reference, the normal stress, σ, and the pore-air pressure, u_a, are expressed relative to the pore water pressure (i.e., are expressed as $(\sigma-u_w)$ and (u_a-u_w)). To achieve this, (43) is written as:

$$a_w=1-a_a-a_c-a_s-\Delta a_s \quad (44)$$

Substituting (44) into (42) and continuing the derivation:

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}-a_a\frac{\partial u_a}{\partial y}\\&-(1-a_a-a_c-a_s-\Delta a_s)\frac{\partial u_w}{\partial y}\\&+a_cf^*\frac{\partial(u_a-u_w)}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+n_s\rho_sg\\&-F_{sy}^w-F_{sy}^a+a_c(u_a-u_w)\frac{\partial f^*}{\partial y}=0\end{aligned} \quad (45)$$

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}-a_a\frac{\partial u_a}{\partial y}-\frac{\partial u_w}{\partial y}+a_a\frac{\partial u_w}{\partial y}\\&+a_c\frac{\partial u_w}{\partial y}+a_s\frac{\partial u_w}{\partial y}+\Delta a_s\frac{\partial u_w}{\partial y}\\&+a_cf^*\frac{\partial(u_a-u_w)}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+n_s\rho_sg\\&-F_{sy}^w-F_{sy}^a+a_c(u_a-u_w)\frac{\partial f^*}{\partial y}=0\end{aligned} \quad (46)$$

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y-\partial u_w}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}-a_a\left(\frac{\partial u_a-\partial u_w}{\partial y}\right)\\&+a_cf^*\frac{\partial(u_a-u_w)}{\partial y}+(a_c+a_s+\Delta a_s)\frac{\partial u_w}{\partial y}\\&+n_s\rho_sg-F_{sy}^w-F_{sy}^a+a_c(u_a-u_w)\frac{\partial f^*}{\partial y}=0\end{aligned} \quad (47)$$

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial(\sigma_y-u_w)}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}-a_a\frac{\partial(u_a-u_w)}{\partial y}\\&+a_cf^*\frac{\partial(u_a-u_w)}{\partial y}+(a_c+a_s+\Delta a_s)\frac{\partial u_w}{\partial y}\\&+n-F_{sy}^w-F_{sy}^a+a_c(u_a-u_w)\frac{\partial f^*}{\partial y}=0\end{aligned} \quad (48)$$

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial(\sigma_y-u_w)}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\\&-(a_a-a_cf^*)\frac{\partial(u_a-u_w)}{\partial y}\\&+(a_c+a_s+\Delta a_s)\frac{\partial u_w}{\partial y}+n_s\rho_sg\\&-F_{sy}^w-F_{sy}^a+a_c(u_a-u_w)\frac{\partial f^*}{\partial y}=0\end{aligned} \quad (49)$$

It is assumed that the cross-sectional area filled with air per gross cross-sectional area of the soil element, a_a, is a function of the porosity with respect to the air phase, n_a:

$$a_a = f\left(n_a\right) \tag{50}$$

Also, it is assumed that the cross-sectional area filled with contractile skin per gross cross-sectional area of the soil element, a_c, is a function of the porosity with respect to the contractile skin, n_c:

$$a_c = f\left(n_c\right) \tag{51}$$

Substituting (50) and (51) into (49):

$$\begin{aligned}&\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial\left(\sigma_y-u_w\right)}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\\&-\left(f\left(n_a\right)-f\left(n_c\right)f^*\right)\frac{\partial\left(u_a-u_w\right)}{\partial y}\\&+\left(f\left(n_c\right)+a_s+\Delta a_s\right)\frac{\partial u_w}{\partial y}+n_s\rho_s g\\&-F_{sy}^w-F_{sy}^a+f\left(n_c\right)\left(u_a-u_w\right)\frac{\partial f^*}{\partial y}=0\end{aligned} \tag{52}$$

The volume of contractile skin is small and therefore:

$$n_c \approx 0 \tag{53}$$

Therefore:

$$f\left(n_c\right) \approx 0 \tag{54}$$

Substituting (54) into (52):

$$\begin{aligned}&\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial\left(\sigma_y-u_w\right)}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\\&-f\left(n_a\right)\frac{\partial\left(u_a-u_w\right)}{\partial y}+\left(a_s+\Delta a_s\right)\frac{\partial u_w}{\partial y}\\&+n_s\rho_s g-F_{sy}^w-F_{sy}^a=0\end{aligned} \tag{55}$$

From [34][35]:

$$a_s \approx 0 \tag{56}$$

Substituting (56) into (55):

$$\begin{aligned}&\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial\left(\sigma_y-u_w\right)}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\\&-f\left(n_a\right)\frac{\partial\left(u_a-u_w\right)}{\partial y}+\Delta a_s\frac{\partial u_w}{\partial y}\\&+n_s\rho_s g-F_{sy}^w-F_{sy}^a=0\end{aligned} \tag{57}$$

For unsaturated soils with incompressible solids, the change in the cross-sectional area filled with solids per gross cross-sectional area of the soil element due to the compressibility of the soil solids, Δa_s is zero. Therefore:

$$\Delta a_s = 0 \tag{58}$$

Substituting (58) into (57) results in the following relationship:

$$\begin{aligned}&\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial\left(\sigma_y-u_w\right)}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\\&-f\left(n_a\right)\frac{\partial\left(u_a-u_w\right)}{\partial y}+n_s\rho_s g-F_{sy}^w-F_{sy}^a=0\end{aligned} \tag{59}$$

Derivation in the *x*-Direction

The derivation in the *x*-direction is performed to obtain the equilibrium equation in the *x*-direction. The equilibrium equation for the soil structure in the *x*-direction is equivalent to the equilibrium equation for the soil structure in the *y*-direction but without the contribution of the gravity force (i.e., the term $\left(\rho-n_a\rho_a-n_w\rho_w-n_c\rho_c\right)g$).

The equilibrium equation for the water phase in the *x*-direction is:

$$\begin{aligned}&-a_w\left(u_w+\frac{\partial u_w}{\partial x}dx\right)dy.dz+a_w u_w dy.dz\\&-F_{sx}^w dx.dy.dz-F_{cx}^w dx.dy.dz=0\end{aligned} \tag{60}$$

where a_w is the cross-sectional area filled with water per gross cross-sectional area of the soil element. Continuing the derivation in (60), the equilibrium for the water phase in the *x*-direction is:

$$\left(a_w\frac{\partial u_w}{\partial x}+F_{sx}^w+F_{cx}^w\right)dx.dy.dz=0 \tag{61}$$

For the air phase, the sum of air forces in the *x*-direction = 0 uses the following relation:

$$\begin{aligned}&-a_a\left(u_a+\frac{\partial u_a}{\partial x}dx\right)dy.dz+a_a u_a dy.dz\\&-F_{sx}^a dx.dy.dz-F_{cx}^a dx.dy.dz=0\end{aligned} \tag{62}$$

Continuing the derivation in (62), the equilibrium equation for the air phase in the *x*-direction is:

$$\left(a_a\frac{\partial u_a}{\partial x}+F_{sx}^a+F_{cx}^a\right)dx.dy.dz=0 \tag{63}$$

For the contractile skin phase, the sum of contractile skin forces in the *x*-direction = 0 uses the following relation:

$$a_c\begin{pmatrix}\sigma_x^c+\dfrac{\partial f^*}{\partial x}\left(u_a-u_w\right)dx\\+f^*\dfrac{\partial\left(u_a-u_w\right)}{\partial x}dx\end{pmatrix}dy.dz-a_c\sigma_x^c dy.dz \qquad (64)$$
$$+F_{cx}^w dx.dy.dz+F_{cx}^a dx.dy.dz=0$$

Continuing the derivation in (64), the equilibrium equation for the contractile skin in the x-direction is:

$$\begin{pmatrix}-a_c\left(u_a-u_w\right)\dfrac{\partial f^*}{\partial x}\\-a_c f^*\dfrac{\partial\left(u_a-u_w\right)}{\partial x}\\-F_{cx}^w-F_{cx}^a\end{pmatrix}dx.dy.dz=0 \qquad (65)$$

Using (61), (63), (65) and equivalent to (29), considering that in equilibrium the sum of forces in the x-direction is equal to zero, the equilibrium equation for an unsaturated soil element in the x-direction is:

$$\frac{\partial\sigma_x}{\partial x}-a_a\frac{\partial u_a}{\partial x}-a_w\frac{\partial u_w}{\partial x}+a_c f^*\frac{\partial\left(u_a-u_w\right)}{\partial x}$$
$$+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}-F_{sx}^w-F_{sx}^a \qquad (66)$$
$$+a_c\left(u_a-u_w\right)\frac{\partial f^*}{\partial x}=0$$

where σ_x is the total normal stress in the x-direction, τ_{yx} is the shear stress on the y-plane in the x-direction, τ_{zx} is the shear stress on the z-plane in the x-direction, F_{sx}^w is the interaction force (i.e., body force) between the water phase and the soil solids in the x-direction, F_{sx}^a is the interaction force (i.e., body force) between the air phase and the soil solids in the x-direction.

The normal stress, σ and the pore-air pressure, u_a are expressed relative to the pore water pressure (i.e., are expressed as ($\sigma - u_w$) and ($u_a - u_w$)). To achieve this, following the procedure in (45), substituting (44) into (66):

$$\frac{\partial\sigma_x}{\partial x}-a_a\frac{\partial u_a}{\partial x}-\left(1-a_a-a_c-a_s-\Delta a_s\right)\frac{\partial u_w}{\partial x}$$
$$+a_c f^*\frac{\partial\left(u_a-u_w\right)}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}-F_{sx}^w-F_{sx}^a \qquad (67)$$
$$+a_c\left(u_a-u_w\right)\frac{\partial f^*}{\partial x}=0$$

Using the same derivation method from (45) to (57):

$$\frac{\partial\left(\sigma_x-u_w\right)}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}-f\left(n_a\right)\frac{\partial\left(u_a-u_w\right)}{\partial x}$$
$$+\Delta a_s\frac{\partial u_w}{\partial x}-F_{sx}^w-F_{sx}^a=0 \qquad (68)$$

For unsaturated soils with incompressible solids, the change in the cross-sectional area filled with solids per gross cross-sectional area of the soil element due to the compressibility of the soil solids, Δa_s is zero. Substituting (58) into (68) results in the following relationship:

$$\frac{\partial\left(\sigma_x-u_w\right)}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}-f\left(n_a\right)\frac{\partial\left(u_a-u_w\right)}{\partial x}$$
$$-F_{sx}^w-F_{sx}^a=0 \qquad (69)$$

Derivation in the *z*-Direction

The derivation in the z-direction is performed to obtain the equilibrium equation in the z-direction. As in the derivation in the x-direction, the equilibrium equation for the soil structure in the z-direction is also equivalent to the equilibrium equation for the soil structure in the x-direction.

The equilibrium equation for the water phase in the z-direction is:

$$-a_w\left(u_w+\frac{\partial u_w}{\partial z}dz\right)dx.dy+a_w u_w dx.dy \qquad (70)$$
$$-F_{sz}^w dx.dy.dz-F_{cz}^w dx.dy.dz=0$$

where a_w is the cross-sectional area filled with water per gross cross-sectional area of the soil element. Continuing the derivation in (70), the equilibrium for the water phase in the z-direction is:

$$\left(a_w\frac{\partial u_w}{\partial z}+F_{sz}^w+F_{cz}^w\right)dx.dy.dz=0 \qquad (71)$$

For the air phase, the sum of air forces in the z-direction = 0 uses the following relation:

$$-a_a\left(u_a+\frac{\partial u_a}{\partial z}dz\right)dx.dy+a_a u_a dx.dy \qquad (72)$$
$$-F_{sz}^a dx.dy.dz-F_{cz}^a dx.dy.dz=0$$

Continuing the derivation in (72), the equilibrium equation for the air phase in the z-direction is:

$$\left(a_a\frac{\partial u_a}{\partial z}+F_{sz}^a+F_{cz}^a\right)dx.dy.dz=0 \qquad (73)$$

For the contractile skin phase, the sum of contractile skin forces in the z-direction = 0 uses the following relation:

$$a_c\begin{pmatrix}\sigma_z^c+\dfrac{\partial f^*}{\partial z}(u_a-u_w)dz\\ +f^*\dfrac{\partial(u_a-u_w)}{\partial z}dz\end{pmatrix}dx.dy-a_c\sigma_z^c dx.dy \\ +F_{cz}^w dx.dy.dz+F_{cz}^a dx.dy.dz=0 \tag{74}$$

Continuing the derivation in (74), the equilibrium equation for the contractile skin in the *z*-direction is:

$$\begin{pmatrix}-a_c(u_a-u_w)\dfrac{\partial f^*}{\partial z}\\ -a_c f^*\dfrac{\partial(u_a-u_w)}{\partial z}\\ -F_{cz}^w-F_{cz}^a\end{pmatrix}dx.dy.dz=0 \tag{75}$$

Using (71), (73), (75) and equivalent to (29), considering that in equilibrium the sum of forces in the *z*-direction is equal to zero, the equilibrium equation for an unsaturated soil element in the *z*-direction is:

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\sigma_z}{\partial z}-a_a\frac{\partial u_a}{\partial z}-a_w\frac{\partial u_w}{\partial z} \\ +a_c f^*\frac{\partial(u_a-u_w)}{\partial z}-F_{sz}^w-F_{sz}^a \\ +a_c(u_a-u_w)\frac{\partial f^*}{\partial z}=0 \tag{76}$$

where τ_{xz} is the shear strength on the *x*-plane in the *z*-direction, τ_{yz} is the shear strength on the *y*-plane in the *z*-direction, σ_z is the total normal stress in the *z*-direction, F_{sz}^w is the interaction force (i.e., body force) between the water phase and the soil solids in the *z*-direction, and F_{sz}^a is the interaction force (i.e., body force) between the air phase and the soil solids in the *z*-direction.

The normal stress, σ, and the pore-air pressure, u_a, are expressed relative to the pore water pressure (i.e., are expressed as ($\sigma - u_w$) and ($u_a - u_w$)). To achieve this, following the procedure in (45), substituting (44) into (76) and following the same derivation method from (45) to (57):

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial(\sigma_z-u_w)}{\partial z} \\ -f(n_a)\frac{\partial(u_a-u_w)}{\partial z}+\Delta a_s\frac{\partial u_w}{\partial z}-F_{sz}^w-F_{sz}^a=0 \tag{77}$$

For unsaturated soils with incompressible solids, the change in the cross-sectional area filled with solids per gross cross-sectional area of the soil element due to the compressibility of the soil solids, Δa_s, is zero. Substituting (58) into (77) results in the following relationship:

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial(\sigma_z-u_w)}{\partial z} \\ -f(n_a)\frac{\partial(u_a-u_w)}{\partial z}-F_{sz}^w-F_{sz}^a=0 \tag{78}$$

For unsaturated soils with compressible solids, (57), (68), and (77) can be written in the following form of stress tensor equation:

$$\begin{bmatrix}(\sigma_x-u_w) & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & (\sigma_y-u_w) & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & (\sigma_z-u_w)\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix} \\ -f(n_a)\begin{bmatrix}(u_a-u_w) & 0 & 0\\ 0 & (u_a-u_w) & 0\\ 0 & 0 & (u_a-u_w)\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix} \\ +\Delta a_s\begin{bmatrix}u_a & 0 & 0\\ 0 & u_a & 0\\ 0 & 0 & u_a\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix}+\begin{Bmatrix}0\\ n_s\rho_s g\\ 0\end{Bmatrix} \\ -\begin{Bmatrix}F_{sx}^w\\ F_{sy}^w\\ F_{sz}^w\end{Bmatrix}-\begin{Bmatrix}F_{sx}^a\\ F_{sy}^a\\ F_{sz}^a\end{Bmatrix}=0 \tag{79}$$

For unsaturated soils with incompressible solids, (59), (69), and (78) can be written in the following form of stress tensor equation:

$$\begin{bmatrix}(\sigma_x-u_w) & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & (\sigma_y-u_w) & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & (\sigma_z-u_w)\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix} \\ -f(n_a)\begin{bmatrix}(u_a-u_w) & 0 & 0\\ 0 & (u_a-u_w) & 0\\ 0 & 0 & (u_a-u_w)\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix} \\ +\begin{Bmatrix}0\\ n_s\rho_s g\\ 0\end{Bmatrix}-\begin{Bmatrix}F_{sx}^w\\ F_{sy}^w\\ F_{sz}^w\end{Bmatrix}-\begin{Bmatrix}F_{sx}^a\\ F_{sy}^a\\ F_{sz}^a\end{Bmatrix}=0 \tag{80}$$

RESULTS

From (79), the stress tensors for unsaturated soils with compressible solids are:

$$\begin{bmatrix}(\sigma_x-u_w) & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & (\sigma_y-u_w) & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & (\sigma_z-u_w)\end{bmatrix} \tag{81}$$

and

$$\begin{bmatrix}(u_a-u_w) & 0 & 0\\ 0 & (u_a-u_w) & 0\\ 0 & 0 & (u_a-u_w)\end{bmatrix} \tag{82}$$

and

$$\begin{bmatrix} u_a & 0 & 0 \\ 0 & u_a & 0 \\ 0 & 0 & u_a \end{bmatrix} \quad (83)$$

From (80), the stress tensors for unsaturated soils with incompressible solids are:

$$\begin{bmatrix} (\sigma_x - u_w) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - u_w) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - u_w) \end{bmatrix} \quad (84)$$

and

$$\begin{bmatrix} (u_a - u_w) & 0 & 0 \\ 0 & (u_a - u_w) & 0 \\ 0 & 0 & (u_a - u_w) \end{bmatrix} \quad (85)$$

From the above derivation, the stress state variables for unsaturated soils with compressible soil solids using the pore-water pressure, u_w, as the stress reference are $(\sigma - u_w)$, $(u_a - u_w)$, and u_w. The stress state variables for unsaturated soils with incompressible soil solids using the pore-water pressure, u_w, as the stress reference are $(\sigma - u_w)$ and $(u_a - u_w)$. Equivalent continuum elements of unsaturated soil with the stress state variables using the pore-water pressure, u_w, as the stress reference are shown in Figure 4(a) for compressible soil solids and Figure 4(b) for incompressible soil solids.

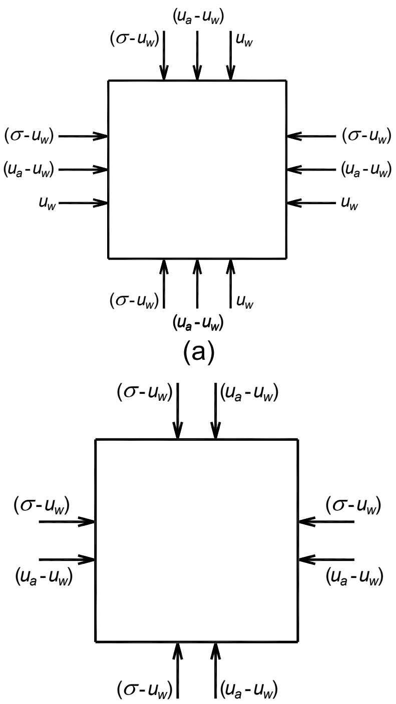


Figure 4. Equivalent Continuum Elements of Unsaturated Soil with the Stress State Variables Using the Pore-Water Pressure, u_w as the Stress Reference: (a) Soil Element with Compressible Soil Solids; (b) Soil Element with Incompressible Soil Solids

DISCUSSION

A detailed comparison between the derivation derived in this study and the previous derivation is presented in Table 1. Tables 1(a) to 1(c) demonstrate the force equilibrium equations for the water phase, air phase, and contractile skin phase, respectively. These comparisons confirm that the results of the derivation in this study are consistent with those found in the literature (i.e., in [1][11]). Table 1(d) higlights the critical difference regarding the elimination of u_w for incompressible soil solids. It shows that the 'fifth term' in the proposed derivation, Δa_s, is equal to zero.

Table 1. Comparison of the Derivation in This Study and the Existing Derivation: (a) Force Equilibrium of Water Phase; (b) Force Equilibrium of Air Phase; (c) Force Equilibrium of Contractile Skin Phase; (d) Elimination of u_w for Incompressible Soil Solids

(a)

Direction	Force Equilibrium in the Existing Derivation	Force Equilibrium in the New Derivation (This Study)
y-axis	$-n_w\left(u_w + \frac{\partial u_w}{\partial y}dy\right)dx.dz + n_w u_w dx.dz$ (in (13))	$-a_w\left(u_w + \frac{\partial u_w}{\partial y}dy\right)dx.dz + a_w u_w dx.dz$ (in (31))
x-axis	$-n_w\left(u_w + \frac{\partial u_w}{\partial x}dx\right)dy.dz + n_w u_w dy.dz$ (in (21))	$-a_w\left(u_w + \frac{\partial u_w}{\partial x}dx\right)dy.dz + a_w u_w dy.dz$ (in (60))
z-axis	$-n_w\left(u_w + \frac{\partial u_w}{\partial z}dz\right)dx.dy + n_w u_w dx.dy$ (in (25))	$-a_w\left(u_w + \frac{\partial u_w}{\partial z}dz\right)dx.dy + a_w u_w dx.dy$ (in (70))

(b)

Direction	Force Equilibrium in the Existing Derivation	Force Equilibrium in the New Derivation (This Study)
y-axis	$-n_a\left(u_a + \frac{\partial u_a}{\partial y}dy\right)dx.dz + n_a u_a dx.dz$ (in (15))	$-a_a\left(u_a + \frac{\partial u_a}{\partial y}dy\right)dx.dz + a_a u_a dx.dz$ (in (33))
x-axis	$-n_a\left(u_a + \frac{\partial u_a}{\partial x}dx\right)dy.dz + n_a u_a dy.dz$ (in (22))	$-a_a\left(u_a + \frac{\partial u_a}{\partial x}dx\right)dy.dz + a_a u_a dy.dz$ (in (62))
z-axis	$-n_a\left(u_a + \frac{\partial u_a}{\partial z}dz\right)dx.dy + n_a u_a dx.dy$ (in (26))	$-a_a\left(u_a + \frac{\partial u_a}{\partial z}dz\right)dx.dy + a_a u_a dx.dy$ (in (72))

(c)

Direction	Force Equilibrium in the Existing Derivation	Force Equilibrium in the New Derivation (This Study)
y-axis	$n_c\left(\begin{matrix}\sigma_y^c \\ +\dfrac{\partial f^*}{\partial y}(u_a-u_w)dy \\ +f^*\dfrac{\partial(u_a-u_w)}{\partial y}dy\end{matrix}\right)dx.dz$ $-n_c\sigma_y^c dx.dz$ (in (17))	$a_c\left(\begin{matrix}\sigma_y^c \\ +\dfrac{\partial f^*}{\partial y}(u_a-u_w)dy \\ +f^*\dfrac{\partial(u_a-u_w)}{\partial y}dy\end{matrix}\right)dx.dz$ $-a_c\sigma_y^c dx.dz$ (in (35))
x-axis	$n_c\left(\begin{matrix}\sigma_x^c \\ +\dfrac{\partial f^*}{\partial x}(u_a-u_w)dx \\ +f^*\dfrac{\partial(u_a-u_w)}{\partial x}dx\end{matrix}\right)dy.dz$ $-n_c\sigma_x^c dy.dz$ (in (23))	$a_c\left(\begin{matrix}\sigma_x^c \\ +\dfrac{\partial f^*}{\partial x}(u_a-u_w)dx \\ +f^*\dfrac{\partial(u_a-u_w)}{\partial x}dx\end{matrix}\right)dy.dz$ $-a_c\sigma_x^c dy.dz$ (in (64))
z-axis	$n_c\left(\begin{matrix}\sigma_z^c \\ +\dfrac{\partial f^*}{\partial z}(u_a-u_w)dz \\ +f^*\dfrac{\partial(u_a-u_w)}{\partial z}dz\end{matrix}\right)dx.dy$ $-n_c\sigma_z^c dx.dy$ (in (27))	$a_c\left(\begin{matrix}\sigma_z^c \\ +\dfrac{\partial f^*}{\partial z}(u_a-u_w)dz \\ +f^*\dfrac{\partial(u_a-u_w)}{\partial z}dz\end{matrix}\right)dx.dy$ $-a_c\sigma_z^c dx.dy$ (in (74))

(d)

Direction	The Fifth Term of the Equation in Existing Derivation	The Fifth Term of the Equation in the New Derivation (This Study)
y-axis	$(n_c+n_s)\dfrac{\partial u_w}{\partial y}$ (in (19))	$\Delta a_s\dfrac{\partial u_w}{\partial y}$ (in (57))
x-axis	$(n_c+n_s)\dfrac{\partial u_w}{\partial x}$ (in (24))	$\Delta a_s\dfrac{\partial u_w}{\partial x}$ (in (68))
z-axis	$(n_c+n_s)\dfrac{\partial u_w}{\partial z}$ (in (28))	$\Delta a_s\dfrac{\partial u_w}{\partial z}$ (in (77))

The stress state variables using the pore-water pressure, u_w, as the stress reference have a common stress reference with the stress state variable for saturated soils (i.e., the effective stress, $(\sigma - u_w)$). This gives an easier understanding for the transition of the stress state variable(s) from unsaturated soils to saturated soils.

The theoretical revisit in this paper offers clearer interpretation and a more comprehensible derivation. In addition, it supports the validity and correctness of the commonly adopted stress state variables used in the constitutive modelling of unsaturated soils. If this is ignored, the basis of the derivation of the shear strength and volume change is questionable.

The stress state variables using u_w as the stress reference derived in this study, $(\sigma - u_w)$ and $(u_a - u_w)$, can be used to calculate the factor of safety, FS. The results should be the same as the calculated FS using the stress state variables using u_a as the stress reference, $(\sigma - u_a)$ and $(u_a - u_w)$, since both yield the same FS.

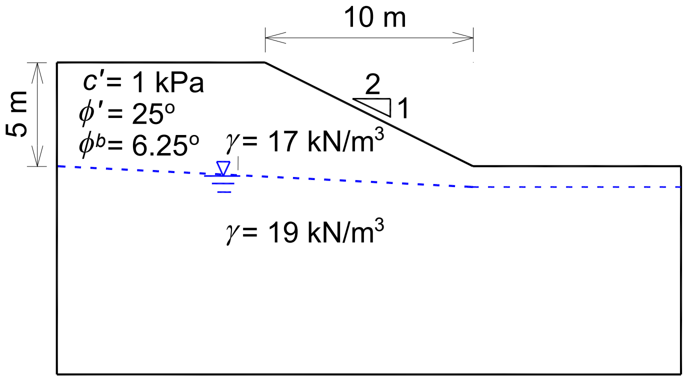
A hypothetical example of calculating FS of a slope is provided in Figure 5 (a). Two cases were analyzed. The calculations were performed numerically using Geo-Slope software [36]. The Bishop's method [37] was used in the calculation of FS. Case 1 is the calculation of FS using the stress state variables available in the software (i.e., $(\sigma - u_a)$ and $(u_a - u_w)$) as in Figure 5(b). The stresses in soil elements (e.g., in Figure 5(c) were used in the calculation of case 2 (Figure 5(c)). Case 2 is the calculation of FS using the stress state variables formulated in this study (i.e., $(\sigma - u_w)$ and $(u_a - u_w)$). In the calculation of case 2, the ϕ'' angle was obtained using the following relationship:

$$\tan\phi'' = \tan\phi^b - \tan\phi' \tag{86}$$

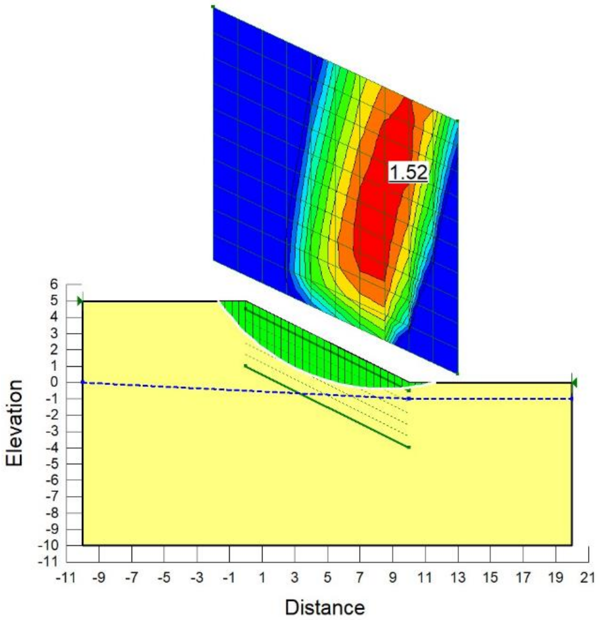
The results are summarized in Table 2. The similarity shows that both stress combinations of variables of unsaturated soils give the same factor of safety, FS. This shows that both stress state variables (i.e., $(\sigma - u_w)$, $(u_a - u_w)$ and $(\sigma - u_w)$, $(u_a - u_w)$) serve as the stress state derivation and contribute to the completeness of the fundamentals of theoretical unsaturated soil mechanics. Also, the method of calculation of FS can be used in the calculation of the factor of safety of a slope during a rainfall-event as in (e.g., in the case of [38][39]).

Table 2. Results of Calculation of Factor of Safety, FS

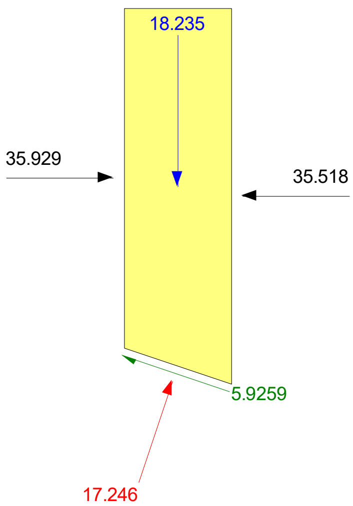
Cases	Stress State Variables	Factor of Safety, FS
1	$(\sigma - u_a)$ and $(u_a - u_w)$	1.52
2	$(\sigma - u_w)$ and $(u_a - u_w)$ (formulated in this study)	1.52



(a)



(b)



(c)

Figure 5. Example of Application in the Calculation of Factor of Safety: (a) Geometry and Soil Parameters; (b) Factor of Safety Calculated Using the Stress State Variables ($\sigma - u_a$) and ($u_a - u_w$); (c) Stresses in the 15th Slice of 30 Slices

CONCLUSIONS

The derivation that explains the stress state variables for unsaturated soils with compressible and incompressible soil solids results in the stress state variables for unsaturated soils. A new approach on the effect of compressibility is used. This is to ensure the correctness of the obtained stress state variables.

The derivation of force equilibrium used in this study uses the term $-a_w\left(u_w+\frac{\partial u_w}{\partial y}dy\right)dx.dz$ $+a_w u_w dx.dz$ for the equilibrium of water in the *y*-direction. This is easier to follow than the term $-n_w\left(u_w+\frac{\partial u_w}{\partial y}dy\right)dx.dz+n_w u_w dx.dz$ used in the existing derivation. The same occurs in the *x*- and *z*- directions.

For the force equilibrium of air in the *y*-direction, the derivation of equilibrium used in this study uses the term $-a_a\left(u_a+\frac{\partial u_a}{\partial y}dy\right)dx.dz$ $+a_a u_a dx.dz$. This is easier to follow than the term $-n_a\left(u_a+\frac{\partial u_a}{\partial y}dy\right)dx.dz+n_a u_a dx.dz$ used in the existing derivation. The same occurs in the *x*- and *z*- directions.

In the derivation of the force equilibrium of contractile skin in the *y*- direction, the term $a_c\left(\sigma_y^c+\frac{\partial f^*}{\partial y}(u_a-u_w)dy+f^*\frac{\partial(u_a-u_w)}{\partial y}dy\right)dx.dz$ is used in this study. This is easier to follow than the term $n_c\left(\sigma_y^c+\frac{\partial f^*}{\partial y}(u_a-u_w)dy+f^*\frac{\partial(u_a-u_w)}{\partial y}dy\right)dx.dz$ used in the existing derivation. The same occurs in the *x*- and *z*- directions.

The result of the derivation in the *y*-direction that shows the effect of compressibility of soil solids is the fifth term of (57): $\Delta a_s(\partial u_w/\partial y)$. This term is zero as the soil solids are incompressible, resulting in the stress state variables ($\sigma - u_w$) and ($u_a - u_w$). This term clearly better describes the compressibility of the soil solids as compared to the term in the existing derivation: $(n_c+n_s)\frac{\partial u_w}{\partial y}$. The same occurs in the *x*- and *z*- directions.

For the stress combination using the pore-water pressure, u_w, as the reference, the stress state variables for unsaturated soils with compressible soil solids are: ($\sigma - u_w$), ($u_a - u_w$), and u_w. The stress state variables for unsaturated soils with incompressible soil solids are: ($\sigma - u_a$) and ($u_a - u_w$). While this is the same as in the previous derivation, the derivation performed in this study provides a better proof for the stress state variables. The revisit and derivation in this study support the validity and correctness of the commonly adopted stress state variables used in the constitutive modelling of unsaturated soils. If this is ignored, the basis of the derivation of the shear strength and volume change are questionable.

The application of the stress state using pore-water pressure, u_w, as the reference is explained using the extended Mohr-Coulomb shear strength envelope equation using the stress state variables ($\sigma - u_w$) and ($u_a - u_w$). A calculation of the slope factor of safety, *FS,* was performed. The result is the same as that using the other stress state variables (i.e., ($\sigma - u_a$) and ($u_a - u_w$)), indicating the compatibility of the derivation.

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